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The prediction of contrails as a problem of diffusion

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Introduction

1. The problem of the formation of contrails which has been the subject of numerous investigations (1) is becoming of increasing importance as the speed and heights at which military aircraft operate increase, but while it is generally accepted that such trails are due to the condensation of water vapour from the engine exhaust no method has been developed which will enable the length, nature and persistence of contrails to be predicted from meteorological observations.

The resemblance between photographs of contrails and photographs of smoke trails from factory chimneys suggested that the solution of the continuous point source problem which had proved fruitful in discussions of the latter problem might afford a possible line of attack upon the former and more difficult problem, and in the present paper the various solutions which have from time to time been proposed for the point source problem are briefly reviewed.

Recent observations of contrails made by the Meteorological Research Flight, Farnborough, in conjunction with the Central Photographic Establishment, Benson, are discussed by means of the most promising of these solutions and the results appear sufficiently encouraging to warrant further study.

The point source solutions

2. Roberts (2) on the basis of the classical theory of turbulence obtained the following solution for the continuous point source problem

$$\chi = \frac{Q}{4\pi K (x^2+y^2+z^2)^{1/2}} \exp - \frac{(y^2+z^2)}{(x^2+y^2+z^2)^{1/2}} \frac{U}{4K} \qquad 2.00$$

where the origin of the cartesian coordinate system is taken at the point source with x measured down wind, y across wind and z vertically upwards, χ is the concentration of smoke at x, y, z, K is the (constant) coefficient of eddy diffusion, u is the mean wind speed (assumed invariant with height) and Q is the rate of emission.

Whilst however first tests of predictions of 2.00 against observations of smoke trails shewed that χ varied directly as Q and inversely as u (which required K to be proportional to the mean wind speed), later observations quoted by Sutton (3) shewed that the peak concentration at y=z=o fell off as $x^{-1.76}$ under adiabatic conditions, whereas 2.00 required that the peak concentration should fall off as x^{-1}.

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The failure of the K = constant concept to give a satisfactory approach to the point source problem was attributed to the non homogeneity of the atmospheric eddy structure, but attempts to modify the classical theory in order to take account of the effect of eddies of many sizes acting together met with little success.

O.G. Sutton (4) discarded the K approach and from an extension of G.I. Taylor's (5) statistical theory of diffusion by continuous movements derived the following approximate solution for the continuous point source problem,

$$\chi = \frac{Q}{\pi C_y C_z x^{2-n}} \exp - \left\{ \frac{1}{x^{2-n}} \left(\frac{y^2}{C_y^2} + \frac{z^2}{C_z^2} \right) \right\} \qquad 2.01$$

where C_y and C_z were generalised coefficients of diffusion across wind and in the vertical respectively and n was an index of the degree of turbulence, but later (6) suggested that

$$\chi = \frac{Q}{4\left[\Gamma\left(\frac{4-n}{2}\right)\right]^2 C_y^{2-n} C_z^{2-n} U x^{2-n}} \exp - \left\{ \frac{1}{x} \left(\frac{y^{\frac{2}{2-n}}}{C_y^2} + \frac{z^{\frac{2}{2-n}}}{C_z^2} \right) \right\} \qquad 2.02$$

gave a better approximation to the observed facts.

Unlike Roberts' solution 2.00, which being derived from a second order differential equation with the appropriate boundary conditions is therefore a unique solution, 2.01 and 2.02 are only two out of an infinite number of possible solutions which can be obtained by the statistical approach. For example if the choice of solution is restricted to a single expression based on the exponential function, Sutton's statistical approach yields

$$\chi = \frac{Q}{4\Gamma\left(1+\frac{1}{r}\right)\Gamma\left(1+\frac{1}{s}\right) C_y^{2/r} C_z^{2/s} U x^{p/r+p/s}} \exp - \left\{ \frac{1}{x^p} \left(\frac{y^r}{C_y^2} + \frac{z^s}{C_z^2} \right) \right\} \qquad 2.03$$

an expression which contains five arbitrary constants p, r, s, C_y and C_z and includes 2.01 and 2.02 as particular solutions.

The reasons for selecting any particular solution of 2.03 as the appropriate solution for the point source problem, and the implications of any particular solution which may be selected, are not clear from the statistical approach which is not concerned with the mechanism of eddy motion. When however 2.03 is derived by means of an extension of the classical theory it is possible to see what assumptions are required and by a combination of theory and observation to decide which solution is likely to give the best representation of the observed facts.

Thus, according to the classical theory, the differential equation which expresses the fact that, when a steady state has been reached, horizontal advection is balanced by lateral and vertical diffusion is

$$U \frac{\partial \chi}{\partial x} = \frac{\partial}{\partial y} \left(K_y \frac{\partial \chi}{\partial y} \right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial \chi}{\partial z} \right) \qquad 2.04$$

/where

where K_y and K_z are the coefficients of eddy diffusion in the y and z directions respectively, and the solution of this equation which fits the boundary conditions appropriate to the point source problem is given by 2.03, provided that

$$K_y = uC_y^2 \frac{p}{r^2} x^{p-1} |y|^{2-r} \qquad 2.05$$

and

$$K_z = uC_z^2 \frac{p}{s^2} x^{p-1} |z|^{2-s} \qquad 2.06$$

Hence 2.01 which is derived from 2.03 by writing p=2-n and r=2 is the solution of 2.04 with

$$K_y = uC_y^2 \left(\frac{2-n}{4}\right) x^{1-n} \qquad 2.07$$

$$K_z = uC_z^2 \left(\frac{2-n}{4}\right) x^{1-n} \qquad 2.08$$

Coefficients of eddy diffusion in the atmosphere which increase with distance downwind from the point at which the smoke is being emitted, are however physically unattractive, since they appear to imply that the atmosphere is aware of the point at which the smoke is being emitted and is able to adjust its eddies accordingly. This consideration would therefore appear to render unacceptable any solution of the type 2.03 with the possible exception of those for which p = 1 (which includes 2.02 as a special case).

With p = 1, 2.03 becomes

$$\chi = \frac{Q}{4\Gamma\left(1+\frac{1}{r}\right)\Gamma\left(1+\frac{1}{s}\right)C_y^{2/r}C_z^{2/s}ux^{1/r+1/s}} \exp\left\{\frac{1}{x}\left(\frac{y^r}{C_y^2}+\frac{z^s}{C_z^2}\right)\right\} \qquad 2.09$$

whilst

$$K_y = \frac{uC_y^2}{r^2} |y|^{2-r} \qquad 2.10$$

and

$$K_z = \frac{uC_z^2}{s^2} |z|^{2-s} \qquad 2.11$$

so that in this case the coefficients of eddy diffusion are independent of the distance downwind from the point at which the smoke is being emitted, but are dependent upon the distance from the axis.

This at first sight seems nearly as unattractive as the former concept, but as a smoke trail may be regarded as an aggregate of a very large number of smoke puffs of various ages, then, focusing attention on one particular puff as it spreads downwind and taking axes moving with the mean wind speed, with the origin of coordinates at the centre of gravity of the puff, the functional dependence of K_y and K_z upon y and z respectively may be taken as indicating that as the puff moves downwind and expands it progressively comes under the influence of larger and larger eddies, but that only eddies which are comparable

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with or are smaller than the smoke puff at any instant have any effect upon its rate of expansion.

A functional form for K_z similar to 2.11 but with u_{10} at a standard height of 10 m. replacing u was derived by the present writer and used to discuss the problems of diffusion of matter (7) from plane sources situated at the earth's surface, and observations near the ground showed that under adiabatic conditions good agreement was reached with a value of $s = 8/7$. If it is permissible to ignore the variation of wind with height near the surface in order to identify 2.09 with observations of smoke trails from point sources near the ground which shew that $\chi \propto x^{-1.76}$, Sutton (3), it follows that with $s = 8/7$ then, to a high degree of approximation, r also $= 8/7$, so that 2.09 reduces to Sutton's solution 2.02 with $n = \frac{1}{4}$.

Since the variation of wind with height is very marked near the ground only the order of magnitude of C_y or C_z can be obtained by comparing 2.02 with observations of smoke trails near the surface and this gives the order of magnitude as $0.30\ cm^{-\frac{1}{4}}$.

In order to prevent possible confusion between generalised diffusion coefficients which in equations 2.01 and 2.02 are of very different dimensions being $(\text{length})^{\frac{1}{8}}$ in the former and $(\text{length})^{1/4}$ in the latter, c_y and c_z will be used for the former and C_y and C_z for the latter. When c.g.s. units are used however the numerical values of the coefficients are approximately the same.

If σ_y is the standard deviation of the smoke particles from the mean at right angles to the wind, then from 2.01

$$\sigma_y^2 = \tfrac{1}{2} c_y^2 x^{2-n} \qquad 2.12$$

whilst from 2.02

$$\sigma_y^2 = \frac{\Gamma\left\{\frac{3}{2}(2-n)\right\}}{\Gamma\left\{\frac{1}{2}(2-n)\right\}} \left(C_y^2 x\right)^{2-n} \qquad 2.13$$

Sutton (4) who tested out 2.12 against Richardson's (8) data on pilot balloons over distances of several hundred kilometres obtained a value of $n = \frac{1}{4}$ in good agreement with the values of n found from observations of smoke trails over short distances near the ground.

From an analysis of the growth of clouds formed by the bursting of anti-aircraft shells between 1 and 5 km., Sutton (4) on the assumption that $c_x = c_y = c_z = c$ also found that $n = \frac{1}{4}$ and that the c's varied between 0.088 and 0.022 $cm.^{\frac{1}{8}}$, the values of c apparently decreasing with height according to a law which yielded $c = 0.30\ cm.^{\frac{1}{8}}$ at a height of 1 m. above the surface and $c = 0$ at a height of 10,000 m. above the surface.

The value of $n = \frac{1}{4}$, but not a law of decrease of c with height, was also found by Sartor and others (9) to fit observations of the growth of smoke puffs between 8 m. and 45 m. R.F. Jones (10) who took observations of smoke puffs at heights varying from 2700 m. to 11700 m. and examined the motion of the smoke puffs at intervals of 20 seconds by means of 2.12 obtained values of n which shewed considerable scatter with however a mean value of .15. The values of σ_x which he obtained were not normally the same as the values of σ_y, but the

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differences were not systematic and the values of c_x and c_y calculated from these values of σ_x and σ_y shewed no variation with height, wind shear or lapse rate. By plotting all values of log c_x and log c_y combined against values of 2-n , he found that the relationship between c and n was given by

$$c = \frac{10^{3.47}}{10^{2.39(2-n)}} \qquad 2.14$$

whence

$$\sigma^2 = 4.4 \times 10^6 \left(\frac{x}{6 \times 10^4}\right)^{2-n} \qquad 2.15$$

With $n = \frac{1}{4}$, 2.14 gives $c = c_x = c_y = 0.195$ cm$^{\frac{1}{8}}$.

Jones' observations when examined by 2.13 yield $C = C_x = C_y = 0.287$ cm$^{\frac{1}{14}}$

The evidence thus available suggests that diffusion from a fixed point source anywhere in the troposphere is capable of being discussed by means of 2.02 with, under adiabatic conditions, $n = \frac{1}{4}$ and probably $C_x = C_y = C_z = 0.30$ cm$^{\frac{1}{14}}$

3. Contrails

(a) General

Aviation fuels are hydrocarbons which upon combustion with the oxygen of the air produce water and heat which are injected into and mix with the wake environment. As at very low temperatures the amount of water vapour which the air can hold in the form of invisible vapour is limited, at such temperatures a proportion of the water which is added appears in the form of visible droplets which then spread downwind and are scattered by turbulent diffusion.

In order to discuss the dispersion of these droplets, it will be assumed (1) that the evaporation of the droplets as they spread downwind can be neglected, and (2) that the turbulent diffusion from a point source moving at u cm. sec^{-1} is the same as that from a fixed point source into a turbulent medium in which the mean wind speed is u cm. sec^{-1}. The limitations of these assumptions are discussed later in sections 4 and 5.

If therefore we take axes moving with the aircraft, with the origin at the exhaust with x measured downwind, y across wind and z in the vertical, then if φ grams of fuel are used each second while cruising at u cm. sec^{-1} and λ grams of water are condensed as a result of the combustion of each gram of fuel, the amount of liquid water at x y z created by the passage of the aircraft is from 2.02

$$\Delta f = \frac{\varphi\lambda}{4\left[\Gamma\left(1+\frac{1}{1+m}\right)\right]^2 u \left(C^2 x\right)^{2/1+m}} \exp - \frac{|y|^{1+m} + |z|^{1+m}}{C^2 x} \qquad 3.00$$

where $C_x = C_y = C$, and $m = \frac{n}{2-n}$.

It is not altogether clear what criterion should be used for defining the visible edge of a contrail, but Roberts (2) in discussing the corresponding problem of a smoke trail assumed that a certain minimum number of particles in a tube of unit cross-section through the observer and the smoke cloud was required in order to obtain obscuration of the background.

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If we follow Roberts' opacity theory, then if N is this number and r and ρ_d are the radius and density respectively of the droplets, assumed spherical,

$$\frac{4}{3}\pi r^3 \rho_d N = \int_{-\infty}^{\infty} \Delta f \, dy \qquad 3.01$$

is the equation for the visible outline of the cloud.

Thus if $|z|$ is the visible radius of the trail at a distance x downwind,

$$\frac{4}{3}\pi r^3 \rho_d N = \frac{q\lambda}{4\left[\Gamma\left(1+\frac{1}{1+m}\right)\right]^2 v (C^2x)^{2/1+m}} \exp -\frac{|z|^{1+m}}{C^2x} \int_{-\infty}^{\infty} \exp -\frac{|y|^{1+m}}{C^2x}\, dy \qquad 3.02$$

$$= \frac{q\lambda}{2\,\Gamma\left(1+\frac{1}{1+m}\right) v (C^2x)^{1/1+m}} \exp -\frac{|z|^{1+m}}{C^2x} \qquad 3.03$$

$$\therefore \frac{|z|^{1+m}}{C^2x} = -\frac{1}{1+m}\log_e x - \log_e\left\{\frac{8}{3} v \Gamma\left(1+\frac{1}{1+m}\right) C^{\frac{2}{1+m}} \pi r^3 \rho_d N\right\} + \log_e(q\lambda) \qquad 3.04$$

If therefore we plot $|z|^{1+m}/x$ against $\log_e x$ with $m = \frac{1}{3}$ (which corresponds to $n = \frac{1}{4}$) we should obtain a straight line if the theory is at all in accord with the facts, and since the slope of the line is $C^2/(1+m)$, C can be determined without any assumptions as to the magnitudes of N, r or λ.

Alternatively if m is an index of the stability of the atmosphere at the level at which the aircraft is flying, then if $|z|^{1+m}/x$ is plotted against $\log_e x$ for different values of m, the points will only lie on a straight line for one particular value of m, so that photographs of contrails made under different conditions of stability appear to afford a possible method of determining m. 3.04 may also be written,

$$\frac{|z|^{1+m}}{x} = \frac{C^2}{(1+m)} \log_e \frac{L}{x} \qquad 3.05$$

where L is the value of x for which $|z|$ vanishes, i.e. L is the length of the visible trail along the axis. It follows therefore that the trail length can be determined from measurements of the trail width at any point downwind if C and m are known and vice versa.

Differentiating 3.05 with respect to x and discarding the modulus we have

$$(1+m)z^m \frac{dz}{dx} = \frac{C^2}{1+m}\left[\log_e \frac{L}{x} - 1\right] \qquad 3.06$$

so that if we consider a trail of given length L, the trail width is a maximum when

$$x = \frac{L}{2.3} \qquad 3.07$$

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i.e. the trail attains its maximum width at about 3/7 of its length independently of C and m, cf. Fig. I which shows a good non-persistent contrail made by a Mosquito of the Meteorological Research Flight and photographed by a Mosquito of the Central Photographic Establishment, Benson.

The corresponding expression for the visible outline of a contrail from 2.01 is

$$\frac{x^2}{C^2 x^{2/(1+m)}} = \frac{1}{1+m} \log_e \frac{L}{x} \qquad 3.08$$

so that observations of the radius of the contrail at specified distances downwind enable a comparison to be made between the two solutions 2.01 and 2.02.

(b) The relation between trail length and drop size

If N is the number of droplets in a tube of unit cross-section through the observer perpendicular to and through the axis of the contrail required to obtain obscuration of the background, then from 3.03

$$\frac{4}{3}\pi r^3 \rho_d N = \frac{\phi\lambda}{2u\Gamma(1+\frac{1}{1+m})(C^2 L)^{1/(1+m)}} \qquad 3.09$$

and in obtaining this relationship no assumptions were made as to the magnitudes of N or r.

An approximate relationship between N and r may be obtained by the following method for which I am indebted to Dr. K.H. Stewart. If the angle subtended by the sun's radius at the earth is Θ radians and the apparent brightness of the sun is B_1 candles cm^{-2}, then the illumination produced by the sun on the contrail is $\pi B_1 \Theta^2$ lumens cm^{-2}. The brightness of the edge of the trail will thus be $(N A B_1 \Theta^2)/4$ candles cm^{-2} where A is the effective scattering cross-section of the drop which, except when the angle of scattering is small, may be assumed to be equal to the geometrical cross-section πr^2.

If now the apparent brightness of the sky is B_2 candles cm^{-2}, the trail will be visible if the contrast between it and its background exceeds 0.02 (which is the value of the threshold of brightness used by Koschmieder (11)), so that N is given by

$$N r^2 = \frac{0.08\, B_2}{\pi B_1 \Theta^2} \qquad 3.11$$

Using the values of the brightness of a clear blue sky $B_2 = 1.2 \times 10^{-1}$ candles cm^{-2}, the brightness of the zenith sun $B_1 = 1.65 \times 10^{5}$ candles cm^{-2}, and

$\Theta = 4.665 \times 10^{-3}$ radians, (these values of B_1, B_2 and Θ are taken from Tables of Physical and Chemical Constants by Kaye and Laby, the value of B_2 at 30,000 feet being reduced from that at the ground in the ratio of the pressures)

$$N r^2 = 8.5 \times 10^{-4} \text{ cm}^2 \qquad 3.12$$

so that from 3.09

$$\frac{4}{3}\pi r \rho_d = \frac{\phi\lambda \times 1.05 \times 10^3}{2u\Gamma(1+\frac{1}{1+m})(C^2 L)^{1/(1+m)}} \qquad 3.13$$

/and

and hence from a knowledge of the trail length and λ we may obtain the order of magnitude of r.

It follows from 3.09 that if L_o is the length of trail for given values of φ_o, u_o and λ_o then the value of L for any other values of these variables is given by

$$L = L_o \left\{ \frac{\varphi \lambda u_o}{\varphi_o \lambda_o u} \right\}^{1+m} \qquad 3.14$$

and since for straight and level flight at normal cruising speed the quantity of fuel used per second is directly proportional to the amount of air by weight which passes through the engine per second, it follows that the lengths of the trails at different levels should be related by

$$L = L_o \left\{ \frac{\lambda \rho}{\lambda_o \rho_o} \right\}^{1+m} \qquad 3.15$$

where ρ_o is the density of the air at the level where $L = L_o$ and $\lambda = \lambda_o$.

(c) The value of λ.

It can be calculated from the chemical reactions involved that for each gram of fuel burned about 1.4 grams of water vapour and about 10,000 calories of heat are produced (1) so that if λ grams appear in the form of visible water for each gram of fuel consumed then clearly $\lambda < 1.4$g.

An approximate expression for λ which is convenient for numerical computation can be readily derived by a slight extension of an argument used in (1). If A is the cross-section of the trail into which the exhaust gases pass at a short distance downwind from the exhaust and it is assumed that all the fuel energy goes as heat into this trail *, then the increase of temperature at this distance downwind is

$$\Delta T = \frac{10{,}000 \times \varphi}{A \rho c_p u} \qquad 3.16$$

and similarly the increase in moisture content of $\Delta\mu$ grams per kilogram is

$$\Delta\mu = \frac{1.4 \times 10^3 \times \varphi}{A \rho u} \qquad 3.17$$

so that the ratio of the moisture increase to the temperature increase is independent of the cross-section of trail affected and is equal to 0.0336 gm kg^{-1} per degree centigrade. This result which was also obtained by Appleman (12) shews that the passage of an aircraft through an initially saturated atmosphere would cause the atmosphere to become supersaturated if $0.0336 > d\mu_s/dT$, where μ_s is the saturated mixing ratio with respect to water corresponding to a temperature T, resulting in the production of water droplets.

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* The percentage of heat which is available for heating the air is usually assumed to be about 80% for piston driven aircraft and about 100% for jet aircraft, so that the values of λ derived in this section may be too low. Provided observations from one type of aircraft only are used to compare theory and observation, errors from this assumption will not be important.

Now since when contrails are present the trail into which the exhaust gases are discharged is always saturated, it follows that the number of grams of water which are condensed for every gram of fuel burned is approximately

$$\lambda = \frac{A\rho U}{q \times 10^3} \left\{ 0.0336 - \frac{d\mu_s}{dT} \right\} \Delta T \qquad 3.18$$

or from 3.16

$$\lambda = \frac{10}{c_p} \left\{ 0.0336 - \frac{d\mu_s}{dT} \right\} \qquad 3.19$$

so that from a knowledge of the variation with temperature of the saturation mixing ratio, λ may be readily calculated from 3.19.

Values of μ_s, the saturation mixing ratio over water, down to temperatures of -50°C, derived from the Goff-Gratch formula which is based upon the integration of the Clausius-Clapeyron equation and on modern experimental data, are given in the Smithsonian Meteorological Tables, but since as pointed out in these Tables, the necessary thermodynamical data for an exact integration of the Clausius-Clapeyron equation do not exist for temperatures below 0°C, the values of μ_s for these negative temperatures are open to doubt. A further difficulty results from the fact that contrails frequently form in the free air at temperatures down to -60°C and no values of μ_s are given for these temperatures. Values of μ_s for temperatures below -50°C have accordingly been derived by using the values of the saturation mixing ratio over ice for these temperatures, given in the Smithsonian Meteorological Tables, multiplied by the ratios, given in (1), of the saturation values of water vapour with respect to a water and ice surface respectively.

The values of λ which have been derived from these values of μ_s and 3.19, and which may accordingly be subject to large errors, are shewn in Fig. II.

In order to simplify the problem it has up to the present been assumed that the contrail consists of water droplets which have formed immediately behind the aircraft in an environment which is saturated with respect to water, and that these droplets spread downwind without change of state. At the levels at which contrails occur the supercooled water droplets should in general freeze very quickly so that the resulting ice crystals would at first grow at the expense of the water environment. As the crystals spread downwind and began to mix with the air of the free environment, which in general is not saturated with respect to ice, they would commence to evaporate. This effect is briefly discussed in the following section.

Evaporation of ice crystals in an unsaturated atmosphere

4. If we assume that the ice crystals are at rest relative to the air then as shewn by Jeffreys (13) the rate of evaporation of an ice crystal of mass M in an unsaturated atmosphere is given by

$$\frac{dM}{dt} = 4\pi C_1 D(\rho_a - \rho_s) \qquad 4.00$$

where ρ_s is the vapour density at the surface of the crystal, ρ_a is the vapour density at a point remote from the crystal, D is the coefficient of diffusion of water vapour in air and C_1 is the electrostatic capacity of the crystal, and this equation has been used to calculate the evaporation times of ice crystals whose shapes approximate to those of conductors of known capacity.

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For a sphere of radius r, $C_1 = r$ and $M = \frac{4}{3}\pi r^3 \rho_i$, where ρ_i is the crystal density, so that 4.00 may be written

$$r\frac{dr}{dt} = \frac{D}{\rho_i}(\rho_a - \rho_s) \qquad 4.01$$

If therefore r is the radius of the crystal at a time t and r_o is the radius of the crystal initially, then

$$r^2 = r_o^2 - \frac{2tD}{\rho_i}(\rho_s - \rho_a) \qquad 4.02$$

It follows therefore that the time taken for a spherical crystal of radius r to evaporate completely is

$$t_1 = \frac{\rho_i r_o^2}{2D(\rho_s - \rho_a)} \qquad 4.03$$

i.e. the drier the environment and the smaller the ice crystal, the more rapidly does it evaporate.

The capacity of an ellipsoidal conductor differs only slightly from that of a spherical conductor of the same volume, and if the axes of the ellipsoid are $2r(1+\alpha)$, $2r(1+\beta)$, $2r(1+\gamma)$ where $(1+\alpha)(1+\beta)(1+\gamma) = 1$, the capacity is approximately

$$C_1 = r\left[1 + \frac{2}{15}(\alpha^2 + \beta^2 + \gamma^2)\right] \qquad 4.04$$

Initially therefore an ice crystal whose shape is ellipsoidal would commence to evaporate at a slightly faster rate than a spherical ice crystal of the same volume. In the case of an ellipsoidal crystal however the evaporation would be greatest at the ends of the major axis and least at the ends of the minor axes and would thus tend to promote spherical symmetry and thereby reduce the differential rate of evaporation, so that the evaporation times of an ellipsoidal crystal should be of the same order as those of a spherical crystal.

For simplicity we shall assume that the ice crystals may be treated as spheres, so that from 4.02 and 4.03

$$\frac{r^2}{r_o^2} = 1 - \frac{t}{t_1} \quad , \quad t_1 > t \qquad 4.05$$

This equation is only valid at some distance from the exhaust when the ice crystals have become widely separated. Now from 3.03 the number of ice crystals lying in a tube of unit cross-section through the observer and perpendicular to and through the axis of the contrail varies as $x^{-1/(1+m)}$, where x is the distance downwind from the aircraft. Hence since the obscuring power varies as the number of crystals and the square of their radii, it follows that if L is the visible length of the trail with particles of radius r_o and L_1 is the length of trail with particles of radius r, then

$$\frac{r^2}{r_o^2} = \frac{L_1^{\frac{1}{1+m}}}{L^{\frac{1}{1+m}}} \qquad 4.06$$

and hence combining 4.05 and 4.06

$$L_1 = L\left\{1 - \frac{t}{t_1}\right\}^{1+m} \qquad 4.07$$

or since the time an ice crystal has been in existence at a distance L_1 downwind from an aircraft moving at u cm. sec^{-1} is L_1/u, then

$$L_1 = L\left\{1 - \frac{L_1}{ut_1}\right\}^{1+n} \qquad 4{\cdot}08$$

5. Comparison with experiments

(a) Length and occurrence of contrail

In a conjoint investigation by the Meteorological Research Flight, Farnborough, in cooperation with the Central Photographic Establishment, Benson, two Mosquito aircraft, one from each unit flew in close proximity, the observer in the M.R.F. aircraft taking observations of the temperature and frost point and the observer in the C.P.E. aircraft taking photographs of the contrails made by the M.R.F. aircraft. The observer in the M.R.F. aircraft also made estimates of the length, intensity and persistence of the contrails made by the C.P.E. aircraft.

Table I gives details of the observations made by the M.R.F. aircraft on March 11th, 1952, whilst Table II gives details of the radii of the M.R.F. contrail at various distances downwind as calculated from photographs made by the C.P.E. aircraft.

The observations were made in the upper troposphere in returning maritime polar air over the Isle of Wight.

During the period of observation, the weather was fine with only small amounts of cumulus cloud of limited vertical extent and with base at 2,500 feet. The winds which were less than 15 kt. at all levels were from WNW between 30,000 and 35,000 feet, and the tropopause which was at 36,000 feet coincided with the maximum heights attained by the aircraft.

Table I

Observations of temperature and frost point on 11th March, 1952

ICAN heights in ft.	Pressure mb.	Temperature °F	Frost Point °F
27000	340	-43°	-55°
28000	329	-48	-55
29000	315	-52	-66
30000	301	-56	-68
31000	287	-62	-71
32000	274	-65	-74
33000	262	-69	-76
34000	250	-73	-80
35000	238	-77	-81
36000	227	-81	-88
38500*	200	-71	-
44500*	150	-64	-
53200*	100	-61	-

* Values taken from the Larkhill Radiosonde Ascent at 1500 G.M.T. on March 11th.

/Table II

Table II

Radius of the trail at specified distances downwind from the aircraft

Radius of trail in m.	Distance downwind from aircraft in m.
1.05	12
1.50	24
1.80	36
1.95	48
2.10	60

Curve A, Fig. III shews the results of plotting $z^{8/7}/x$ against $\log_{10} x$ and it can be seen that as required by 3.05 the points lie on a straight line from whose slope a value of $C_z = 0.26$ is obtained in very good agreement with the values of C_y and C_x derived from Jones' smoke puff observations. Curve B, Fig. III shews the results of plotting $z^2/x^{7/4}$ against $\log_{10} x$ in order to test out 3.08 and it can be seen that in this case the points do not conform to a straight line.

The intercept of curve A with the log x axis gives the visible length of the trail downwind as 215 m. It was unfortunately not possible on this occasion to check this from calculations of the length of the M.R.F. trail from the C.P.E. photographs, as in order to obtain accurate measurements of the trail radius the aircraft flew too close for complete trail lengths in excess of 70 m. to be photographed.

The length of the C.P.E. contrail however as estimated by the observer in the M.R.F. aircraft cf. Table IV was 150 m. which is in reasonably good agreement with, and as expected from 4.08 is lower than, the theoretical value. The effect of the humidity of the environment on the trail length is discussed in 5(c).

Curve a, Fig. II shews the temperature observations for March 11th, 1952, plotted on a height temperature diagram supplemented by temperatures above 36,000 feet taken from the Larkhill ascent on the same day. It can be seen from this curve that if our computations of λ are correct, no liquid water can be condensed below 26,000 feet or above 50,000 feet, so that on this occasion contrails could have been predicted as liable to form at all levels between 27,000 feet and 49,000 feet. It can also be seen from this curve that the value of λ at 30,000 feet is 0.73 g. At 30,000 feet, the aircraft flew at reduced speeds varying from a true airspeed of 235 kt. to 290 kt. while the fuel consumption varied from 80 gallons per hour to 110 gallons per hour, so that q/v varied from 6.71×10^{-3} g. cm^{-1} at the lowest speed to 7.46×10^{-3} g. cm^{-1} at the highest speed. The value of q/v at 30,000 feet may therefore be taken as 7.1×10^{-3} g. cm^{-1} with a tolerance of about $\pm$ 5%.

It follows from 3.00 that the value of δf at the visible edge of the contrail at a distance of 215 m. downwind is

$$\delta f = \frac{7.1 \times 10^{-3} \times 7.3 \times 10^{-1}}{4\left[\Gamma(1\tfrac{7}{8})\right]^2\left[(0.26)^2 \times 21500\right]^{7/4}}$$

$$= 4.14 \times 10^{-3} \text{ g m}^{-3}$$

500

No direct measurements of the amount of solid water in contrails are available to check this, but Goldie (14) and Parker (15) have estimated that the least density of water in ice crystal form required to produce a visible contrail should be of the order of 4×10^{-3} g. m^{-3} in good agreement with this theoretical result.

/Table III

Table III shews the lengths of contrails as computed from 3.15 together with details of the trails as estimated and photographed. Since the values of φ/υ may be in error by about 5% it follows from 3.14 that the trail lengths may be in error by about the same amount due to this cause. The neglect of evaporation of the ice crystals as they spread downwind in an unsaturated atmosphere leads to a slight overestimation of the trail lengths, but as will be shewn later in 5(c) this effect is in general small. The errors in λ are unknown but, from a posteriori considerations, appear to be small.

Table III - Computed and observed contrails 11th March, 1952

Height in feet	λ g.	$\frac{\varphi}{\upsilon}$ x10^{-3} g.sec^{-1}	L computed in m.	Details of MRF trail from photographs	Details of CPE trail as estimated
27000	0.16	7.92	43	weak trail > 67 m.	slight trail 48 m.
28000	0.38	7.64	111	weak trail > 67 m.	slight trail 48 m.
29000	0.55	7.37	162	moderate trail > 67 m.	moderate trail 111 m.
30000	0.73	7.10	215	good trail > 67 m.	moderate trail 150 m.
31000	0.87	6.83	251	good trail > 67 m.	long trail
32000	0.93	6.59	260	good trail > 67 m.	long trail
33000	1.01	6.35	274	good trail > 67 m.	long trail
34000	1.07	6.12	280	good trail > 67 m.	long trail
35000	1.12	5.90	284	good trail > 67 m.	long trail
36000	1.17	5.67	285	good trail > 67 m.	long trail
38500	0.93	5.14	196	No observations	
44500	0.43	4.07	90	No observations	
53200	0	-	0	No observations	

Two other series of double flights made by the M.R.F. and C.P.E. Mosquitos are also available for discussion and these have been plotted as curves b and c in Fig. II.

It can be seen from curve b, which shews the temperatures for March 4th, 1952, that no liquid water can condense below 24,000 ft. or above 34,000 ft. so that on this occasion contrails could have been predicted as liable to form between these limits. These in point of fact materialised and, together with the computed values, are detailed in Table IV.

Table IV - Computed and observed contrails on 4th March, 1952

Height in feet	λ g.	$\frac{\varphi}{\upsilon}$ x10^{-3} g.cm^{-1}	L computed	Details of MRF trail from photographs	Details of CPE trail as estimated
25000	0.30	8.49	97	No trails visible	25 very faint
26000	0.50	8.20	166	60 moderate	25 faint
27000	0.60	7.92	196	90 moderate	39 moderate
28000	0.54	7.64	167	150 moderate	48 moderate
29000	0.70	7.37	214	195 dense	87 moderate
30000	0.60	7.10	172	210 dense	63 moderate
31000	0.57	6.83	157	150 moderate	63 moderate
32000	0.25	6.59	58	intermittent trail	39 moderate
33000	0.20	6.35	42	60 weak	30 moderate
34000	0	6.12	0	No trail visible	No trail visible

The descriptive comments of the contrails made by the C.P.E. aircraft and noted by the M.R.F. observer were very consistent with deductions made from photographs of the contrails made by the M.R.F. aircraft, but for the flights on March 4th, 1952, cf. Table IV the estimated trail lengths were consistently lower than the measured trail lengths.

/This

This may be due to a systematic tendency for visual estimates to be less than measurements from photographs, due to differences in threshold contrasts, or to the difference in view with respect to the sun with consequently very different illumination, or to the trails from the two aircraft being of very different lengths, as on this occasion the photographed M.R.F. aircraft was overtaking the photographing C.P.E. aircraft and was therefore under more power and generating more trail cf. 3.14, or to a combination of all three causes.

It can be seen from curve c which depicts the temperatures for October 24th, 1952, that no liquid water can condense out at any level and hence 'Nil contrails' could have been forecast and none in fact occurred.

Inspection of Fig. II in conjunction with these three ascents suggests that the isopleth $\lambda = 0$ which gives temperature limits below which contrails form and above which contrails are impossible has been correctly derived, and Fig. IV which shows the first appearances of all contrails together with details of the relative humidity with respect to ice of the free environment as recorded by the M.R.F. between the 30th November, 1951, and March 13th, 1953, support this. It can also be seen from Fig. IV that the first appearance of the contrail bears no relationship to the humidity of the environment, a fact which was first noted by Bannon (16).

(b) Crystal size and persistence of contrail

Using the values of $\frac{\psi}{\sigma} = 7.1 \times 10^{-3}$ g. cm^{-1}, $\lambda = 7.3 \times 10^{-1}$ g. and L = 215 m. which were found at 30,000 feet on March 11th, 1952, the radius of the ice crystals, assumed spherical, is from 3.13

$$r = \frac{7{\cdot}1 \times 10^{-3} \times 7{\cdot}3 \times 10^{-1} \times 1{\cdot}05 \times 10^{3}}{\frac{4}{3} \times \frac{22}{7} \times 9{\cdot}2 \times 10^{-1} \times 2 \times \Gamma(1\tfrac{7}{8}) \times [(0{\cdot}26)^{2} \times 21500]}^{7/8}$$

$$= 14\mu \qquad 5{\cdot}01$$

That this is of the right order of magnitude is shewn by laboratory experiments by Mason (17) on the rate of growth of ice crystals in a supersaturated atmosphere which indicated that at temperatures of about -40°C the ice crystals occurred mainly in the form of hexagonal prisms whose equivalent radii were about 13μ to 20μ , and the observations of ice crystals in the single penetration into a long and persistent contrail made by an aircraft of the M.R.F. in general support this. ✕

Using the data of March 11th, 1952, which is given in Table II, the times taken for a spherical crystal of radius 15μ to evaporate completely have been computed from 4.03 and these times together with details of the persistency of the trails noted by the M.R.F. observer are given in Table V.

Table V - Evaporation times and persistency 11th March, 1952

Height	$\rho_s - \rho_a$	t_1 sec.	Persistency = P / Non Persistency = N.P.
27000	.052	110	N.P.
28000	.026	220	N.P.
29000	.034	170	N.P.
30000	.023	240	N.P.
31000	.013	420	N.P.
32000	.011	490	P.
33000	.007	790	P.
34000	.006	920	P.
35000	.003	1850	P.
36000	.004	1400	P.
38500	(.040)	(14)	(N.P.)

/In

✕ Communicated privately by Mr. R.J. Murgatroyd of M.R.F.

In the stratosphere the frost point usually decreases rapidly with height while the temperature either remains steady or increases with height so that in the stratosphere $\rho_s - \rho_a$ increases with height cf. Bannon and others (16). The estimates of $\rho_s - \rho_a$ and of the persistency at 38,500 are typical of conditions in the stratosphere with a well marked Type I tropopause at about 36,000 feet.

Table VI gives the temperatures and frost points for the ascents on March 4th together with the theoretical evaporation times for a spherical crystal of radius 15 μ computed from 4.03 and details of the persistency as noted by the M.R.F.

Table VI - Evaporation times and persistency March 4th, 1952

Height	Temperature °F	Frost Point °F	$\rho_s - \rho_a$	t_1 sec.	Persistency
25000	-44	-56.5	.053	110	N.P.
26000	-48	-68	.054	105	N.P.
27000	-51	-71	.045	125	N.P.
28000	-51	-78	.051	115	N.P.
29000	-55	-83	.041	140	N.P.
30000	-54	-87	.044	130	N.P.
31000	-54	-81.5	.042	135	N.P.
32000	-49	-93	.061	95	N.P.
33000	-49	-94	.065	85	N.P.

Inspection of Tables V and VI suggests that there is a general relationship between these theoretical evaporation times and the persistency, and that for theoretical evaporation times (with ice crystals initially of radius 15 μ) of about 7 minutes or less, contrails are non persistent.

(c) Persistency and trail length

Since in the preceding analysis only qualitative use has been made of the theoretical evaporation times of a spherical ice crystal, it makes no difference whether the times in Tables V and VI are as given, or whether they should be multiplied by some constant factor to take account of the shape effect. In the present section however in which we are concerned with the actual evaporation times, it is necessary to consider whether the shapes in which ice crystals are normally found, i.e. flat hexagonal plates or hexagonal prisms, materially alter the order of magnitude of these times.

The capacities of hexagonal plates and prisms are difficult to calculate, but Mason (17) found that to a first approximation hexagonal plates could be treated as oblate spheroids and hexagonal prisms as prolate spheroids.

It can be seen from 4.04 that the capacities of an oblate spheroid and of a prolate spheroid are approximately the same and if, to get some idea of the magnitude of the shape effect, we take $\alpha = \frac{1}{2}$, $\beta = 0$ and $\gamma = -\frac{1}{3}$, which corresponds to the case of an oblate spheroid in which the major axis is approximately double the minor axis, $C = 1.06\ r$ approximately. Hence a crystal originally in the shape of an oblate spheroid will commence to evaporate slightly faster than a spherical crystal of the same volume, and the evaporation times of the former will be lower than those of the latter by about 3%. It would thus appear that the theoretical evaporation times of a spherical ice crystal may be taken as representing to a first approximation the actual evaporation times of ice crystals of equal volume but of different shapes.

/Table VII

Table VII which has been constructed from 4.08 shows how contrails of various lengths are reduced for specified evaporation times and it can be seen from this table that for contrails below 500 m. in length the evaporation from the ice crystals as they spread downwind has very little effect upon the trail length.

Table VII - Evaporation times and trail lengths

L \ t	∞	400	40[x]
500	500	495	449
250	250	249	236
200	200	199	192
100	100	100	99
50	50	50	50

For trail lengths in excess of some particular value which, in the case of Mosquitos flying under reduced power, is about 250 m. cf. Table III, the predicted lengths based upon an infinite evaporation time are less than those actually observed, or in other words the rate of diffusion of ice crystals at distances in excess of this length is far slower than that predicted by 2.03.

This is probably a consequence of the assumption that the trail from an aircraft moving through the air with a velocity u cm. sec^{-1} is the same as that from a fixed point in the atmosphere in which the mean wind speed is u cm. sec^{-1}. It is clear that this assumption can only be approximately correct for a limited distance downwind, for in the immediate wake of the aircraft we have found that the diffusion coefficients are represented by expressions similar to those found from Jones' smoke puff observations at points remote from the aircraft with however u the velocity of the aircraft relative to the air in the former replacing U the velocity of the air in the latter. It suggests however that the turbulent wake behind a piston type aircraft extends for about 250 metres and that if the turbulence in the wake is sufficient to dissipate the contrail before this length is attained then only non persistent contrails are formed. It follows therefore that a necessary condition for a persistent contrail is $\frac{q\lambda}{U} > 5 \times 10^{-3}$.

Conclusions

A comparison between contrail observations made conjointly by Mosquitos of the Meteorological Research Flight and the Central Photographic Establishment and deductions made from the solution of the continuous point source problem, together with certain simplifying assumptions, indicates

(1) that the lengths and widths of non persistent contrails can be predicted with considerable accuracy from a knowledge of the fuel consumption, speed of aircraft and temperature of the environment alone,

(2) that the formation of contrails is independent of the humidity of the air and depends only upon the temperature of the environment and to a slight extent upon the type of aircraft fuel,

(3) that contrails form anywhere within a zone given by the intersection of the environment curve and the isopleth $\lambda = 0$ and increase in length up to the level of the tropopause, if this is within the zone, and decrease very rapidly with height in the stratosphere

/and

x Unlikely to be encountered

and suggests

(4) that long and persistent trails should only occur at temperatures below -40° in (a) an atmosphere which is supersaturated with respect to ice and for which $\lambda > 0$, so that any droplets which are produced would freeze and grow at the expense of the environment or (b) at very low temperatures where $\frac{q_s}{\rho} > 5 \times 10^{-3}$ together with a sufficiently high relative humidity with respect to ice for the evaporation time of a spherical ice crystal of radius 15μ to be greater than 7 minutes.

A preliminary examination of long and persistent contrails made by Mosquitos of the M.R.F. shews that nearly 90% fall within categories (a) and (b) but the remaining 10%, if the observing and instrumental techniques of humidity measurements by aircraft are correct, indicate that some other factor which has not been taken into account in the foregoing analysis is required.

Acknowledgment is due to Mr. R.J. Murgatroyd, Senior Meteorological Officer, Meteorological Research Flight, Farnborough and Mr. G.W. Hurst, Senior Meteorological Officer, Central Photographic Establishment, Benson for the observations and photographs of contrails on which this report is based.

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M.R.P. 830

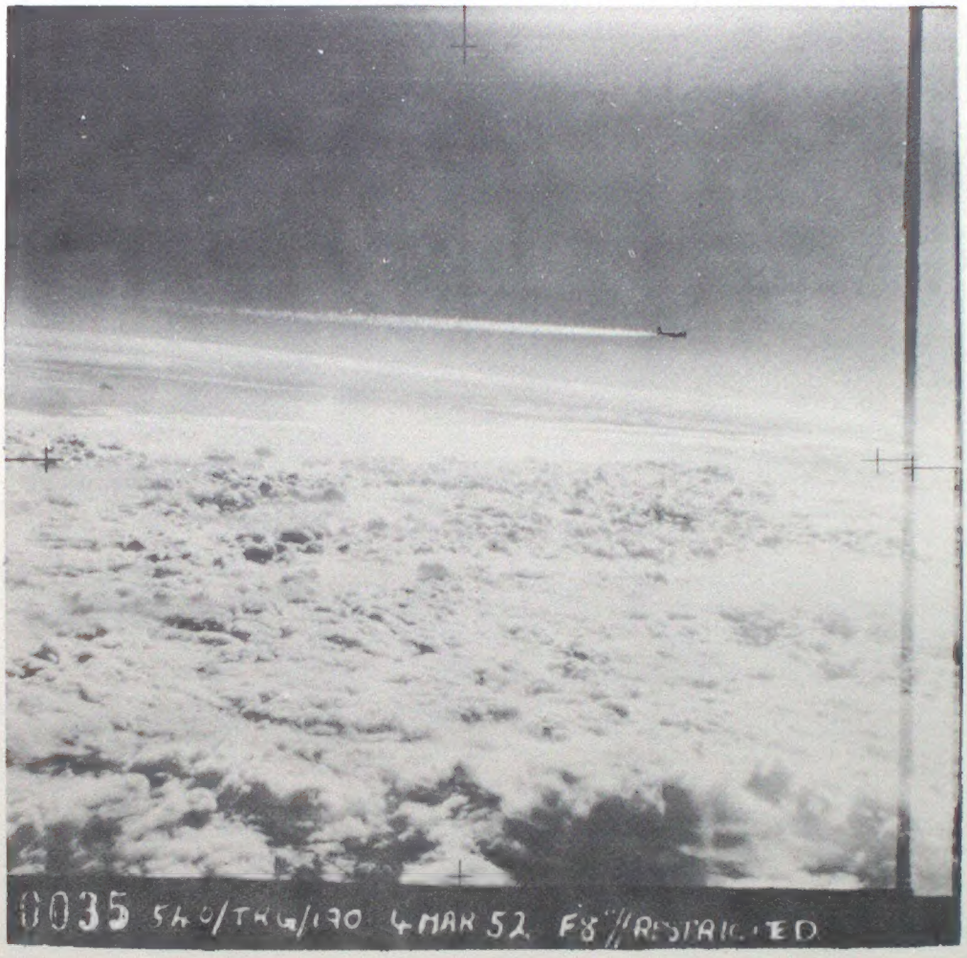

Fig 1. Non-persistent contrail, Larkhill area.

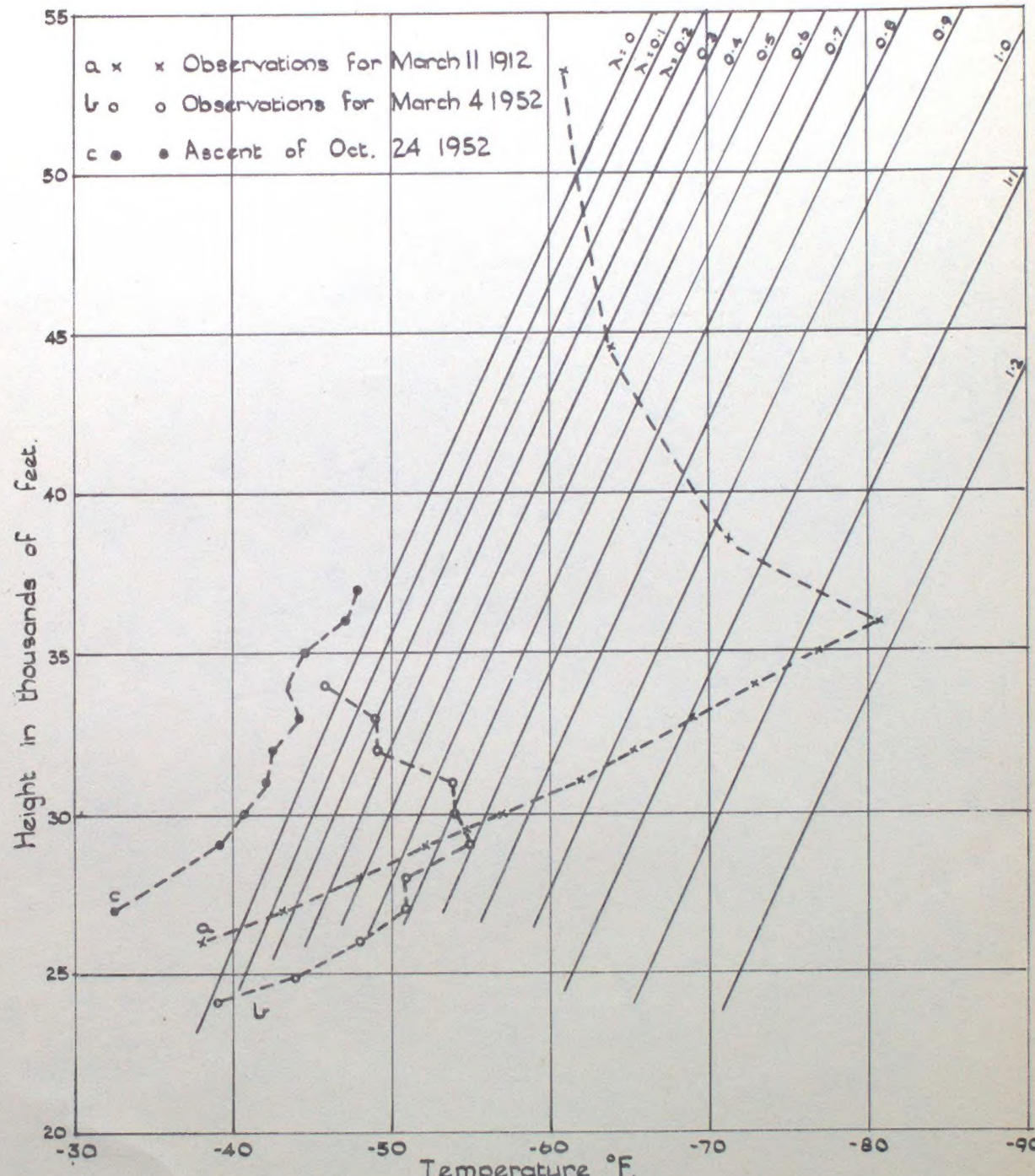

Fig. II. Number of grams of water condensed for each gram of fuel burned.

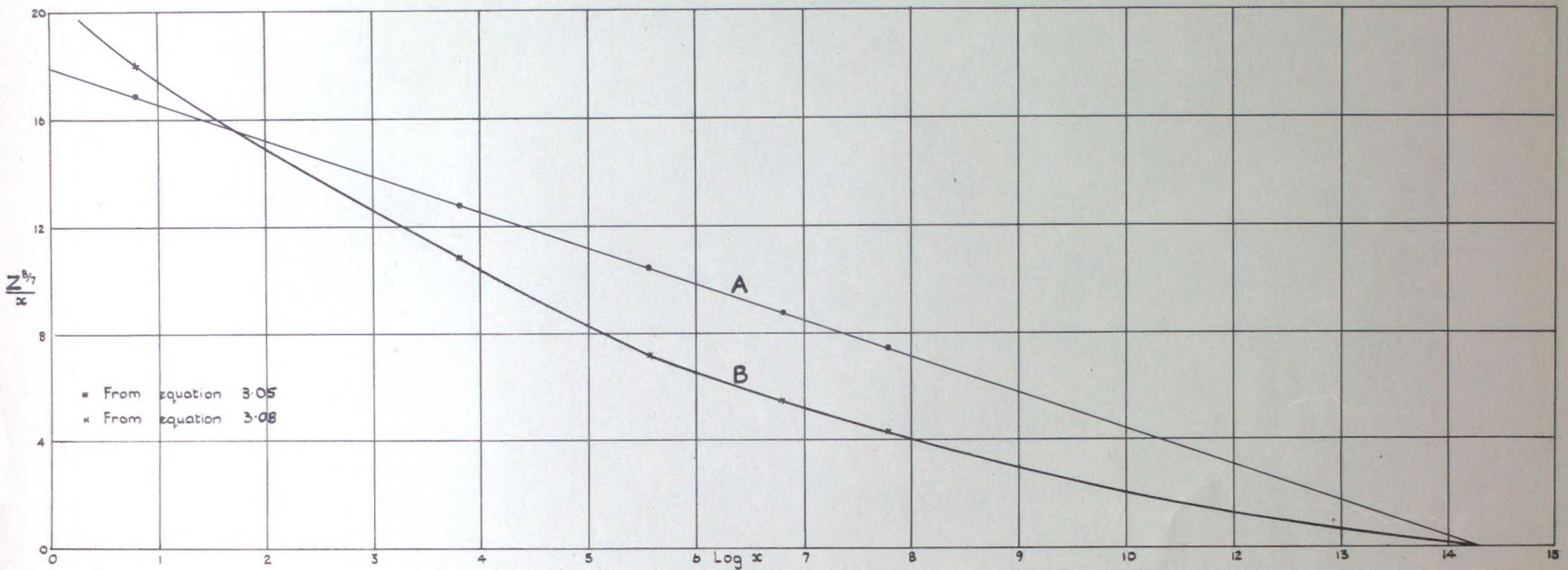

Fig. III. Growth of the visible outline of a contrail.

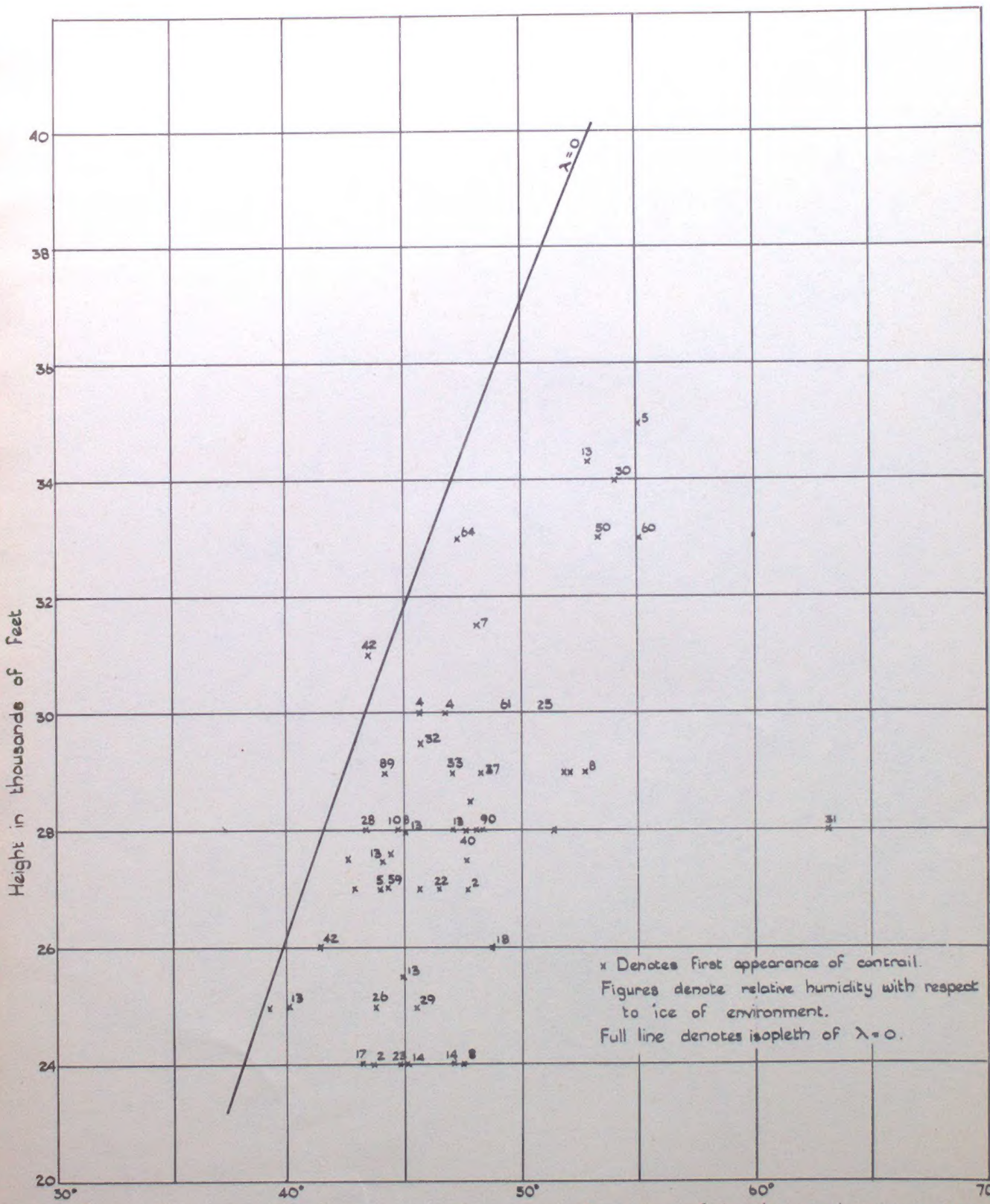

Fig.IV First appearance of a contrail as a function of height and temperature.